$V = P_3$

$U = \{ p \in P_3 \mid p(2) = 0 \}$

$W = \{ p \in P_3 \mid p'(1) = 0 \}$

Find, U, W, $U \cap W$, $U + W$.

Ans:- $V = \{ p \mid \text{degree}(p) \leq 3 \}$

$= \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_0, a_1, a_2, a_3 \in \mathbb{R} \}$

$U = \{ p \in P_3 \mid p(2) = 0 \}$

$= \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_0 + 2a_1 + 4a_2 + 8a_3 = 0 \}$

$= \{ (-2a_1 - 4a_2 - 8a_3) + a_1 x + a_2 x^2 + a_3 x^3 \mid a_1, a_2, a_3 \in \mathbb{R} \}$

$= \{ a_1 (x-2) + a_2 (x^2 - 4) + a_3 (x^3 - 8) \mid a_1, a_2, a_3 \in \mathbb{R} \}$

$\dim(U) = 3$

$W = \{ p \in P_3 \mid p'(1) = 0 \}$

$= \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_1 + 2a_2 + 3a_3 = 0 \}$

$= \{ a_0 + (-2a_2 - 3a_3) x + a_2 x^2 + a_3 x^3 \mid a_0, a_2, a_3 \in \mathbb{R} \}$

$= \{ a_0 + a_2 (x^2 - 2x) + a_3 (x^3 - 3x) \mid a_0, a_2, a_3 \in \mathbb{R} \}$

$\dim(W) = 3$

$U \cap W = \{ p \in P_3 \mid p(2) = 0, \; p'(1) = 0 \}$

$= \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_0 + 2a_1 + 4a_2 + 8a_3 = 0,$
$a_1 + 2a_2 + 3a_3 = 0, \; a_0, a_1, a_2, a_3 \in \mathbb{R} \}$

Now, $a_1 = -2a_2 - 3a_3$

$a_0 + 2(-2a_2 - 3a_3) + 4a_2 + 8a_3 = 0$

$\Rightarrow a_0 - 4a_2 - 6a_3 + 4a_2 + 8a_3 = 0$